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The Primitive Recursive Analysis of Ordinary Differential Equations and the Complexity of Their Solutions

JOHN CLEAVE

School of Mathematics, University of Bristol, England

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1. INTRODUCTION

The subject of our analysis is the well-known Cauchy–Lipschitz existence theorem for solutions of $Dy = f(t, y)$ [*5*]. The task of making the classical theorem constructive has already been effected by Henrici [*4*]. We take the final step of formulating some concepts in the spirit of Goodstein's recursive analysis [*2*] which enable Henrici's proof to be expressed in primitive recursive arithmetic [*1*]. In fact, once the concepts have been framed and certain standard forms defined, writing Henrici's proof in terms of primitive recursive analysis is a trivial but tedious exercise.

Besides noting that the Cauchy–Lipschitz theorem is constructive in a very narrow sense, we exploit Henrici's proof to estimate the computational complexity of the solution of the differential equation in terms of its position in the Grzegorczyck hierarchy of primitive recursive functions [*3*]. We find that for $n \geqslant 3$, if $f \in \mathscr{E}^{(n)}$, then there is a function $y(t, \alpha)$ in $\mathscr{E}^{(n)}$ satisfying $Dy = f(t, y), y_0 = \alpha$. There is no difficulty in extending the primitive recursive analysis to simultaneous coupled differential equations.

2. CONCEPTS OF RECURSIVE ANALYSIS

2.1. *Notation.* The Q, N denote the sets of rational and natural numbers respectively. As in [*2*] a rational number is treated as a triple $(p, q)/r$ of natural numbers, $r > 0$. By using a pairing function a one-one mapping π of Q into N can be constructed. Let f be a function on $N^{(m)} \times Q^{(n)}$ into Q. Then π induces a function $f^* : N^{(n+m)} \to N$; thus

$$f^*(p, q,...; r, s,...) = \pi f(p, q,...; \pi^{-1}(r), \pi^{-1}(s),...).$$

We use this principle to classify functions on the rationals in terms of Grzegorczyck's hierarchy of primitive recursive functions [*3*]. Thus f is said to be in $\mathscr{E}^{(n)}$ or primitive recursive according as f^* is so classified. We note that the simple functions on the rationals such as addition, subtraction, multiplication and division are all in $\mathscr{E}^{(3)}$.

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Conventionally,

$k, l, m, n, p, q, r, s, t$ denote natural numbers,

a, b, d, w, x, y, z denote rationals,

$f, g, h, \ldots$ denote functions into Q,

$F, G, H, \ldots$ denote functions into N.

Rational variables are always separated from natural number variables by a semicolon.

For $n \in N$, $x \in Q$, nx denotes $x + x + \cdots + x$ (n times). $\{m\}$ denotes the rational 2^{-m}, i.e., $(1, 0)/2^m$. The $Q(p)$ denotes the set of rationals $\pm n\{p\}$ (i.e., triples of natural numbers of the form $(n, 0)/2^p$ or $(0, n)/2^p$). We then define

$$N(p; x) = \begin{cases} \min k(x \leqslant (k+1)\{p\}) & \text{if } x \geqslant 0 \\ 0 & \text{if } x < 0 \end{cases}$$

$$x_p = \begin{cases} N(p; x)\{p\} & \text{if } x \geqslant 0 \\ -N(p; x)\{p\} & \text{if } x < 0 \end{cases}$$

Note (1). If $x > 0$, x_p is the largest multiple of 2^{-p} (i.e., member of $Q(p)$) less that x. Thus for $n > 0$

$$x_p = N(p; x)\{p\} < x \leqslant (N(p; x) + 1)\{p\} = x_p + \{p\}.$$

(2) $N(p; x)$ and x_p (i.e., function of x and p) are in $\mathscr{E}^{(3)}$.

2.2. *Recursive continuity and differentiability.* The basic technique of Goodstein's recursive analysis is, in the first instance, to restrict functions $F(x)$ of a real variable to rational arguments so that for each rational r, $F(r) = \lim_{p\to\infty} r_p$ where $(r_0, r_1, \ldots)$ is a Cauchy sequence of rationals. Next define a function $f(p; r)$ on $N \times Q$ by: $f(p; r) = r_p$. Thus $F(r) = \lim_{p\to\infty} f(p; r)$. We can then conceive of $F(r)$, r rational, as a convergent sequence $(f(0; r), f(1; r), \ldots)$ of rational functions. Definitions of convergence, continuity and differentiability are then framed directly in terms of the function $f(p; r)$.

With the exception of Definition 5, the definitions given below are minor modifications of those given by Goodstein [*2*].

Definition 1. Two functions $f(p; x, \ldots)$, $g(p; x, \ldots)$ with the same number of variables are *equivalent* (written $f \sim g$) if there exists a function $N(k; x, \ldots)$ such that

$$p \geqslant N(k; x, \ldots) \rightarrow |f(p; x, \ldots) - g(p; x, \ldots)| < \{k\}.$$

DEFINITION 2.

(1) A function $f(p; x, y)$ is *convergent, uniformly in* y for $a \leqslant x \leqslant b$, $-d \leqslant y \leqslant +d$ if there exists a function $M(k; x)$ such that for all x, y in these ranges

$$q, p \geqslant M(k; x) \rightarrow |f(q; x, y) - f(p; x, y)| < \{k\}.$$

(2) A function $f(p; x, y)$ is *convergent, totally in* y for $a \leqslant x \leqslant b$ if there exists a function $M(k; d, x)$ such that for all x in this range, all d and all y such that $-d \leqslant y \leqslant +d$,

$$q, p \geqslant M(k; d, x) \rightarrow |f(q; x, y) - f(p; x, y)| < \{k\}.$$

DEFINITION 3.

(1) A function $f(p; x, y)$ is *continuous* in the range $a \leqslant x \leqslant b$, $-d \leqslant y \leqslant +d$ if there exists a strictly increasing function $C(k)$ and a function $D(k; x, y, x^1, y^1)$ such that if $a \leqslant x, x^1 \leqslant b$, $-d \leqslant y, y^1 \leqslant +d$

$$| x - x^1 | + | y - y^1 | \leqslant \{C(k)\} \,\&\, p \geqslant D(k; x, y, x^1, y^1)$$
$$\rightarrow |f(p; x, y) - f(p; x^1, y^1)| < \{k\}.$$

(2) A function $f(p; x, y)$ is *continuous, totally in* y for $a \leqslant x \leqslant b$ if there exists a function $C(k; d)$, increasing in k and a function $D(k; d, x, y, x^1, y^1)$ such that for all x, x^1 in this range, all d and all y, y^1 such that $-d \leqslant y, y^1 \leqslant +d$

$$| x - x^1 | + | y - y^1 | < \{C(k; d)\} \,\&\, p \geqslant D(k; d, x, y, x^1, y^1)$$
$$\rightarrow |f(p; x, y) - f(p; x^1, y^1)| < \{k\}.$$

(Thus if $f(p, x, y)$ is convergent (continuous) totally in y, then it is convergent uniformly in y (continuous, resp.) for y in every finite interval $[-d, +d]$, but in such a way that the modulus M (moduli C, D, resp.) can be primitive recursively computed from d.)

DEFINITION 4. A function $f(p; x)$ is *differentiable* in $[a, b]$ if there exist functions $f^1(p; x)$ (a *derivative* of f), $D(k)$, $C(k; x, y)$ such that for all x, y in $[a, b]$,

$$| x - y | \leqslant \{D(k)\} \,\&\, p \geqslant C(k; x, y)$$
$$\rightarrow |f(p; x) - f(p; y) - (x - y) f^1(p; x)| < | x - y | \{k\}.$$

If f^1 is a derivative of f we write "$D_x f(p; x) \sim f^1(p; x)$".

Definition 5. A function $f(p; x, y)$ satisfies a *Lipschitz condition* in $[a, b]$ if (L): There exists a function S and a number L in Q such that for all x in $[a, b]$

$$p \geqslant S(k; x, y, y^1) \rightarrow |f(p; x, y) - f(p; x, y^1)| < L| y - y^1 | + \{k\}.$$

S is the *Lipschitz modulus of* f.

In terms of these concepts one can prove in primitive recursive arithmetic:

Theorem 1. *If for x in the interval $[a, b]$, $f(p; x, y)$ satisfies a Lipschitz condition, is convergent totally in y, and is continuous totally in y, then there exists a (primitive) recursive function $y(p; x, \alpha)$, continuous totally in α for $a \leqslant x \leqslant b$, satisfying*

$$y(p; a, \alpha) = \alpha$$

$$D_x y(p; x, \alpha) \sim f(p; x, y(p; x, \alpha)).$$

Moreover, if for some $n \geqslant 3$, f, its Lipschitz modulus and its moduli of convergence and continuity are in $\mathscr{E}^{(n)}$, then so is y.

A proof of the first statement of this theorem—a primitive recursive version of the classical existence theorem—will not be given here as it results from a more or less routine application of the principles of recursive analysis as developed by Goodstein [*2*] to the proof of the existence theorem so clearly presented in the first chapter of Henrici's book [*4*].

The proof of the second assertion of theorem 1—the estimate of complexity in terms of the Grzegorzcyck Hierarchy [*3*]—necessitates a more delicate scrutiny of certain constructions occurring in the existence proof; it will be given in detail in Section 3.

Before embarking on this proof we note that many of Goodstein's results are achieved by using the recursive invariance of such properties as convergence and continuity, i.e., if P is such a property, then for any functions f, g

$$P(f) \,\&\, f \sim g \rightarrow P(g)$$

This permits simple standard forms of functions with these properties to be constructed. For the purpose of formalising the Cauchy–Lipschitz theorem we state (without proof) the existence of a suitable standard form in Theorem 2 and the recursive invariance in Theorem 3 below.

First, adapting and extending Th. 2.2 of [*2*] the following uniformity theorem can be proved using Goodstein's techniques (Clause (5) is of importance only for the estimate of complexity proved in Section 3).

Theorem 2. *If for x in $[a, b]$ $f(p; x, y)$ satisfies a Lipschitz condition and is continuous totally in y then there exists a function g such that*

(1) $f \sim g$

(2) $p \geqslant q \rightarrow |g(p; x, y) - g(q; x, y)| < \{q\}$

(3) *there exists a function C such that if* $a \leqslant x, x^1 \leqslant b, -d \leqslant y, y^1 \leqslant +d$ *then* $|x - x^1| + |y - y^1| \leqslant \{C(k; d)\} \rightarrow |g(k; x, y) - g(k; x^1, y^1)| < \{k\}$.

(4) *there exists a rational number L and a function U such that*

(i) $|g(p; x, y) - g(p; x, 0)| < L|y| + \{p\}$.

(ii) $-d \leqslant y, y^1 \leqslant +d \,\&\, p \geqslant U(k; d)$
$\rightarrow |g(p; x, y) - g(p; x, y^1)| < L|y - y^1| + \{k\}$.

(5) *for all* p, y *and all* x *in* $[a, b]$, $g(p; x, y) \in Q(p + 3)$

Moreover if for $n \geqslant 3$ f *and its moduli are in* $\mathscr{E}^{(n)}$, *so are* g, C, U.

Then, using some results from Ch. 2, 3 of [2] we have

THEOREM 3. *If* $f(p; x, y) \sim g(p; x, y)$, f *is convergent totally in* y *and continuous totally in* y *for* $a \leqslant x \leqslant b$, $y(p; x)$ *is differentiable and* $D_x y(p; x) \sim f(p; x, y(p; x))$ *for* $a \leqslant x \leqslant b$, *then* $D_x y(p; x) \sim g(p; x, y(p; x))$ *for* x *in this range.*

It follows from these two theorems that if f satisfies the conditions of Theorem 3, together with a Lipschitz condition, then $D_x y \sim f(p; x, y)$ can be solved by taking an equivalent function g in the standard form provided by Theorem 2.

Finally, we note that Theorem 1 can easily be extended to initial value problems with non-rational boundary conditions. For the proof of Theorem 1 actually constructs functions $D(k; d)$, $C(k; d, x, y)$ such that

$$|\alpha| < d \,\&\, |x - x'| \leqslant \{D(k; d)\} \,\&\, p \geqslant C(k; d, x, y)$$
$$\rightarrow |y(p; x, \alpha) - y(p; x', \alpha) - (x - x')f(p; x, y(p; x, \alpha))| < |x - x'|\{k\}.$$

Hence if $\alpha(p)$ is a primitive recursive real number (Cauchy sequence) in Goodstein's sense [2], a bound d on $|\alpha(p)|$ can be computed and so there exists primitive recursive functions D', C' such that

$$|x - x'| \leqslant \{D'(k)\} \,\&\, p \geqslant C'(k; x, y)$$
$$\rightarrow |y(p; x, \alpha(p)) - y(p; x', \alpha(p)) - (x - x')f(p; x, y(p; x, \alpha(p)))$$
$$< |x - x'|\{k\}.$$

Hence

$$D_x y(p; x, \alpha(p)) \sim f(p; x, y(p; x, \alpha(p)))$$
$$y(p; a, \alpha(p)) = \alpha(p).$$

In this sense, $y(p; x, \alpha(p))$ is the solution of the differential equation whose initial value is the real number $\alpha(p)$.

With this preparation one can easily formalise Henrici's proof.

3. Complexity of the Solution of the Differential Equation

Without loss of generality we can assume that $a = 0$, $b = 1$. The construction of a solution is based upon Euler's method of obtaining an approximate solution. The interval $[0, 1]$ is divided into subintervals of length 2^{-p}. Then the numbers Z_n, $0 \leqslant n \leqslant 2^p$, are computed by the recursion

$$\begin{aligned} Z_0^{(p)} &= \eta \\ Z_{n+1}^{(p)} &= Z_n^{(p)} + 2^{-p} f(n2^{-p}, Z_n^{(p)}). \end{aligned} \tag{1}$$

From this an Euler polygon is constructed by joining the points $(n2^{-p}, Z_n^{(p)})$ to $((n+1)\,2^{-p}, Z_{n+1}^{(p)})$, thus obtaining a function $y^{(p)}(x)$. The existence proof then shows that $\lim_{p\to\infty} y^{(p)}(x)$ exists and is a solution of $dy/dx = f(x, y)$, $y(0) = \eta$.

In our proof of Theorem 1 we can, by Theorems 2, 3, take a function g equivalent to f and satisfying the uniformity conditions of Theorem 2. We shall assume, then that g, and its moduli, C, U are in $\mathscr{E}^{(n)}$, $n \geqslant 3$. Moreover, we may also ensure that

$$x \notin [0, 1] \rightarrow g(p; x, y) = 0. \tag{2}$$

Then, corresponding to the classical procedure (1) we define for all n,

$$\begin{aligned} Z(p, 0; \eta) &= \eta \\ Z(p, n+1; \eta) &= Z(p, n; \eta) + \{p\} g(p; n\{p\}, Z(p, n; \eta)). \end{aligned} \tag{3}$$

Note that by (2), $Z(p, n; \eta) = Z(p, 2^p; \eta)$ if $n > 2^p$. Euler lines are now defined by

$$y(p; x, \eta) = \begin{cases} Z(p, N(p; x); \eta) + (x - x_p) g(p; x_p, Z(p, N(p; x); \eta)) & \text{if } x \in [0, 1], \\ 0 & \text{if } x \notin [0, 1]. \end{cases}$$

(see Fig. 1). Equation (3) enables Z^* to be defined recursively: since $+$, 2^p are $\mathscr{E}^{(3)}$ functions, and $g \in \mathscr{E}^{(n)}$, there exists a function H in $\mathscr{E}^{(n)}$ such that

$$\begin{aligned} Z^*(p, o, m) &= m \\ Z^*(p, n+1, m) &= H(p, n, Z^*(p, n, m)). \end{aligned} \tag{4}$$

Clearly, if $Z \in \mathscr{E}^{(n)}$, $n \geqslant 3$ then $y \in \mathscr{E}^{(n)}$. Hence to estimate the complexity of the solution we have to classify Z (precisely, Z^*) and its modulus of convergence. Henrici (page 19 [4]) shows that there exists a number $c > 0$ such that

$$| Z_n^{(p)} | \leqslant | \eta | \exp L + \frac{c}{L} (\exp L - 1). \tag{5}$$

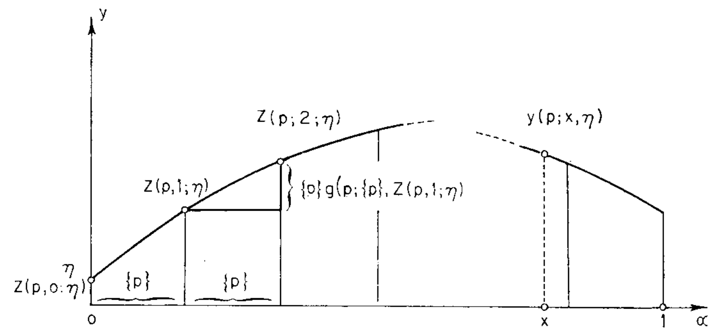

FIG. 1. Construction of the Euler line $y(p; x, \eta)$.

Now Goodstein (page 96 [2]) defines the exponential function by

$$\exp(0; x) = 1, \qquad \exp(n + 1; x) = \exp(n; x) + x^{n+1}/(n + 1).$$

But from pages 96–97 of [2] it follows that

$$e(x) > |\exp(n; x)|, \tag{6}$$

where

$$e(x) = ((2N(x))^{2N(x)}, 0)/(2N(x))!,$$
$$N(x) = \min n((n, 0)/1 \geqslant |x|).$$

Then $e \in \mathscr{E}^{(3)}$. Now $|\eta| e(L) + C/L(e(L) - 1) \equiv d(\eta)$ is also a function in $\mathscr{E}^{(3)}$. Hence by (5), following Henrici's proof of (5) we can obtain a primitive recursive proof of

$$|Z(p, n; \eta)| \leqslant d(\eta). \tag{7}$$

Next, following exactly Henrici's Lemma 1.3, p. 21 [4], and its application on p. 22, we can construct a function $F(k; \eta)$ and a proof of:

$$x \in [0, 1] \;\&\; q > p \geqslant F(k; \eta) \rightarrow |y(p; x, \eta) - y(q; x, \eta)| < \{k\}.$$

Thus y converges. A close analysis shows that F is constructed from d, the moduli C, U, and certain $\mathscr{E}^{(3)}$ functions by the operations of composition and bounded minimum. Now the classes $\mathscr{E}^{(n)}$ are closed under these operations (Th. 4.6 [3]) and $C, U, d \in \mathscr{E}^{(n)}$, $n \geqslant 3$. Hence $F \in \mathscr{E}^{(n)}$.

The final step of placing Z^* in $\mathscr{E}^{(n)}$ (and hence $y \in \mathscr{E}^{(n)}$) demands a closer analysis of the recursion (3). Basically, our aim will be to show that a consequence of (7) is that the recursion (4) is limited in the sense of [3]. From this it follows that $Z \in \mathscr{E}^{(n)}$.

We commence this task by noting that so far we have not defined precisely how addition and subtraction of rationals, considered as triples of natural numbers as on page 2, is to be effected: There are, in fact, several ways of doing this. But by the assumption that g satisfies the conditions of Theorem 2—in particular condition (5) which demands that $g(p; x, y) \in Q(p + 3)$, i.e., $g(p; x, y)$ is an integral multiple of $2^{-(p+3)}$—it is easy to see that the addition in the construction of $Z(p, n + 1; \eta)$ from $Z(p, n; \eta)$ as prescribed by (3) can be so carried out that, because of the factor $\{p\}$,

$$n > 0 \rightarrow Z(p, n; \eta) \in Q(p + 4). \tag{8}$$

Next the pairing function used in 2.1 can be so chosen that the function π has the property

$$p^1 \geqslant p \,\&\, q^1 \geqslant q \,\&\, r^1 \geqslant r \rightarrow \pi((p^1, q^1)/r^1) \geqslant \pi((p, q)/r) \tag{9}$$

and so

LEMMA 4. *For all p and all x, y in $Q(p)$*

$$|x| < y \rightarrow \pi(x) < \max(\pi(y), \pi(-y)).$$

Proof. Suppose $y = (m, 0)/2^p$. Then

(i) if $x = (n, 0)/2^p$ then $|x| = x$ and so as $|x| < y$ we have $n < m$. Hence by (9)

$$\pi(x) = \pi((n, 0)/2^p) < \pi((m, 0)/2^p) = \pi(y). \tag{10}$$

(ii) if $x = (0, n)/2^p$ then $|x| = (n, 0)/2^p$; thus, as $|x| < y$, $n < m$. Hence by (9)

$$\pi(x) = \pi((0, n)/2^p) < \pi((0, m)/2^p) = \pi(-y). \tag{11}$$

By (10), (11), $\pi(x) < \max(\pi(y), \pi(-y))$.

COROLLARY 5. *For $n > 0$*

$$\pi Z(p, n; \eta) < M(p; \eta),$$

where

$$M(p; \eta) = \max(\pi((M^1(p; \eta), 0)/2^{p+4}), \pi((0, M^1(p; \eta))/2^{p+4})),$$

and

$$M^1(p; \eta) = N(p + 4; d(\eta)) + 2.$$

Proof. As $d(\eta) > 0$ we have by 2.1. Note (1)

$$d(\eta) < (N(p + 4; d(\eta)) + 2)\{p + 4\} = (M^1(p; \eta), 0)/2^{p+4}.$$

Hence by (7)

$$|Z(p, n; \eta)| < (M^1(p; \eta), 0)/2^{p+4}.$$

The corollary now follows from Lemma 4

COROLLARY 6. *For* $n > 0$,

$$Z^*(p, n, m) < M(p; \pi^{-1}m).$$

It is clear that as $d \in \mathscr{E}^{(n)}$, so is M. Hence by Corollary 6 and (4), Z^* is defined by limited recursion from functions in $\mathscr{E}^{(n)}$. Thus $Z^* \in \mathscr{E}^{(n)}$. This concludes the proof that $y(p; x, n) \in \mathscr{E}^{(n)}$.

Problem. The function 2^p plays an essential role in Henrici's proof as formalised above because the Euler lines are constructed by successive halving of the steplength. But 2^p is a quite complicated function, being in $\mathscr{E}^{(3)}$ but not in $\mathscr{E}^{(2)}$—hence the condition $n \geqslant 3$ in Theorem 1. It would seem, then, that to decrease the lower limit on n would necessitate a quite different construction from that given here. Can the number 3 in Theorem 1 be replaced by a number < 3?

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